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Maria Chalkou
PhD Mathematics, National and Kapodistrian University of Athens

THE CONTRIBUTION OF THE PUBLICATION OF 2 MANUSCRIPTS OF THE 15TH AND THE 18TH C. TO THE SCIENCE OF THE HISTORY OF MATHEMATICS

Abstract. The paper concerns topics of arithmetic, algebra and geometry which are included in the publication of the Codex Vindobonensis phil. Gr. 65 ff. (11r-126r) of the 15th c. (Chalkou, 2006), and of the manuscript 72 of the 18th c. of the historical Library of Demetsana (Chalkou, 2009). During the study of the 2 manuscripts the interest mainly focused on the mathematical analysis of the methods of the authors, and their significance in the development of the History of Mathematics. The paper also aims to highlight the necessity of easier and broader access to the Sources of Cultural Heritage and the value of digitizing its archives.

We attempt to briefly describe the time the 2 codes were written, the language, the influences and the mathematical fields which comprise their content.

We make known the findings which consolidated the view that the Byzantine manuscripts the Mathematical Encyclopedia of the Byzantines, while the manuscript 72 of the 18th c. is one of the first texts with non-elementary Mathematics during Ottoman rule, and it includes Euclidean Geometry by Nikephoros Theotokes, topics of algebra but also the commercial Mathematics of the Byzantines. From the 'Mathemataria' which were found in the School of Demetsana it is evident that the students were taught, among other things, theoretical and practical arithmetic as well as Euclidean geometry from the manuscript 72, which covered syllabus of today's junior and senior high school. The School was considered higher, and certain manuscripts which were found in its library contain material which is basic but of university level.

The anonymous author of Codex 65 writes that his main source is the work of Greek scholars, and that he has been influenced by the Hindus, the Chinese and the Persians through the Latin scholars due to the commercial transactions between the Byzantines and the West.

In the Codex 65we discovered categories of problems whose solution is achieved through methods unknown to this day. During the research we studied reliable sources of the History of Mathematics in which no data related to these methods were found (Loria and Kovaios, 1972) and (Heath, 1921) and (Smith, 1958).

Then certain methods led us to formulate and prove new mathematical propositions in the field of number theory.

Keywords: Byzantine Mathematics, number theory, Mathematics of the 18th c. in the Greek schools, geometry by Nikephoros Theotokes

1. Introduction

During the study and the publication of the 2 manuscripts the method which we followed was related to the particular groups into which we categorized their problems and mainly concerned the ways of solving the mathematical issues which are contained in them. The methods of solution were explored regarding their historical origin as well as their development to this day.

Then in order to reach the publication of the manuscripts, we extended our scientific research of many years into fields of the history and the development of the

Byzantine Empire and next into fields of the enslaved Greece until the last years of the Ottoman Rule.

We also searched for the common practices during the solution of the problems contained in the two manuscripts. The common problems as well as the common practices in their solution led us to strong indications that Mathematics was uninterruptedly taught in Greece.[1]

Codex 65 is anonymous, in paper form, and the part of pages pp. (11r-126r) of the Codex dates back to the 15th c., specifically 1436 or even earlier (Chalkou, 2006). The whole codex was obtained by Augerious von Busbeck, when he was ambassador of Ferdinand I in the court of sultan Suleiman II (1555-1562 AD). The preamble and the first two chapters, that is pp. 1r-10v, were published by J. I. Heiberg (Hunger, 1994), while in 1963 Hunger published the third part of the codex, that is pp. 126v-140r. Then Hunger with his team attempted to publish the greater part of the codex, pp. (11r-126r). This attempt did not succeed, and in 1978 the team altogether abandoned the endeavour.

However, in 1997, on Professor Pantelis Karelos's request, I received by post in microfilm from the National Library of Austria pp. (11r-126r) to study them and publish them. The content of the microfilm was printed by the competent office of the Greek Parliament in 230 densly-written pages A3 size (Figure 1). It was published in 2006 by the Aristotle University of Thessaloniki and was titled Tractatus Mathematical Vindobonensis Graecus (Ελληνική Βιενναία Μαθηματική Πραγματεία), or (ΕλλΒιενΜαθΠραγμ.).[2]

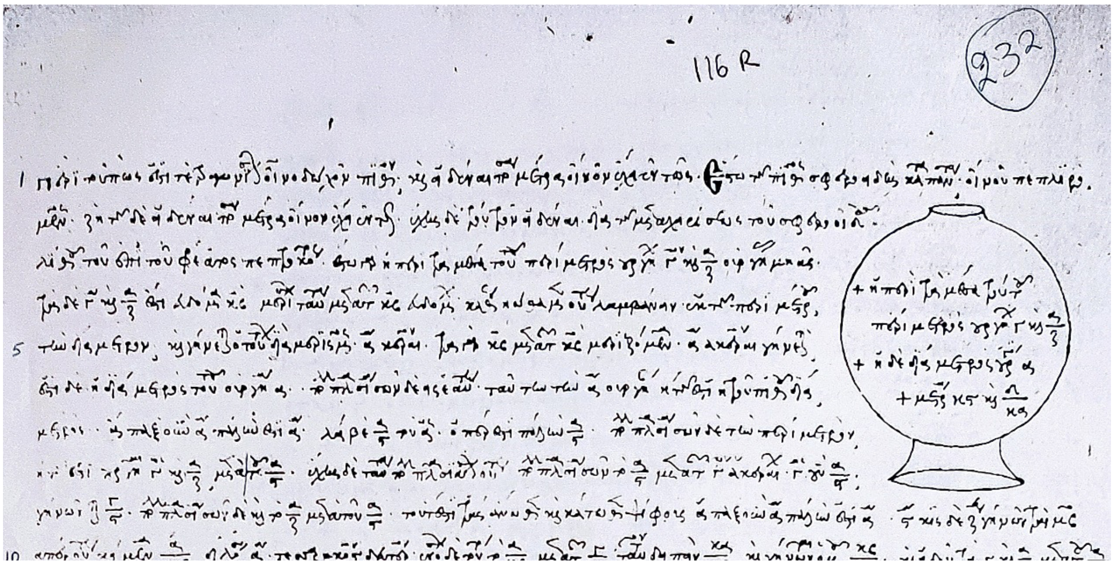

Figure 1. Geometry problem inΕλλΒιενΜαθΠραγμ.

[1]As an example we mention the operations between fractions which are carried out with the same methods as today's (Hunger and Vogel, 1963).

[2]The first publication of the Tractatus Mathematical Vindobonensis Graecus (ΕλλΒιενΜαθΠραγμ.), was made in the form of a doctoral dissertation by the NKUA in 2003 and the second one in the form of critical edition, edited by myself (Introduction, transcription, comments) by the Centre for Byzantine Research of the Aristotle University of Thessaloniki, in 2006.

Codex 72 of the Public Historical Library of Demetsana is anonymous and in paper form, was written about the middle of the18th c., and was taught during the end of the Ottoman Rule in the Greek Schools, among which was the school of Demetsana (Chalkou, 2009). In 2006 I visited the Library of Demetsana in order to present them with a copy of the then recently published by the Aristotle University ΕλλΒιενΜαθΠραγμ. (Chalkou, 2006), in which they had expressed an interest. They toured me around the premises of the Library, where totally coincidentally I noticed a geometry notebook of the 18th c. protected in a showcase. The people there allowed me to scrutinize it for as long as was necessary so that on my request a few days later I received by post – to study it and publish it – its content in the form of 250 photographs of its pages, which were in the size of a small notebook in a cd (Figure 2).[3]

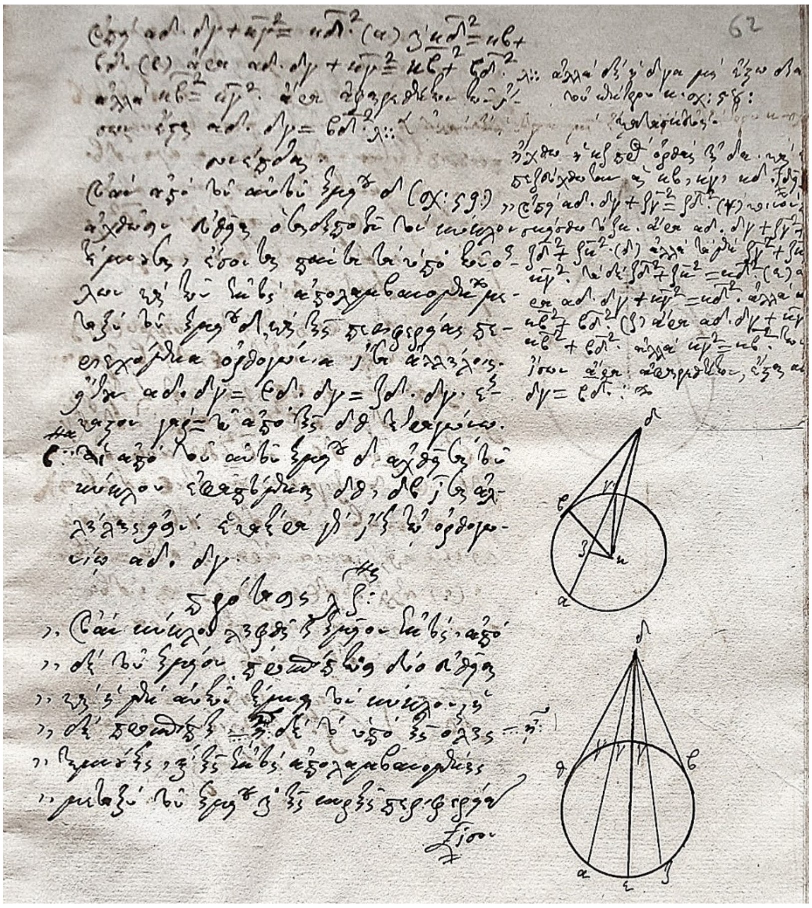

Figure 2. Geometry topics in Codex 72 of the Library of Demetsana

2. Codex 65of the 15th c. pp. 11r-126r

ΕλλΒιενΜαθΠραγμ. contains a book of arithmetic with solved problems, which cover a very broad field of topics appropriate for teaching in today's Primary School as well as Junior and Senior High School. The vast variety of problems makes it difficult to establish the kind of students to whom it is addressed. If it was a comprehensive programme of teaching, then in our view the audience could consist of students of all the grades of today's primary and secondary education, but also of merchants, silversmiths and goldsmiths, craftsmen and individuals who were going to be state

[3]The critical edition (Introduction, transcription, comments) of the Codex 72was printed in the form of self-publication in 2009.

officials. Apart from the kind of problems, particularly interesting is the mathematical terminology which is used and is for the most part unknown to today's mathematicians.

2.1. The language, the influences and the relevant references to the mathematical fields which comprise the content of ΕλλΒιενΜαθΠραγμ. (Chalkou, 2021). The language used in ΕλλΒιενΜαθΠραγμ. is atticized with scientific terms and certain expressions probably of Cypriot origin, such as ‛το βηλάριν την τζόχαν' in order of ‛το βηλάρι', ‛το καντάριν', in order of ‛το καντάρι', ‛το βουτζίν' in order of ‛το βουτσίον'. Its author does not seem to pay particular attention to spelling, as he focuses his interest on the analytical way in which he will effectively teach the issues which he presents. The poetic tone which shows in the flow of the text makes study easier and predisposes the reader to continue reading. The teaching ability of the teacher and his knowledge on methodology lead us to the conclusion that he was a scholar.

The author of ΕλλΒιενΜαθΠραγμ. had studied the ancient Greek mathematicians, he is influenced by Hero of Alexandria, Diophantus, Euclid, Archimedes, and he was informed about the prevailing scientific tendencies in teaching of his time, since he was influenced in many ways by other sources, too. In the first chapter he himself mentions the Persian influence, yet through Latin scholars. The latin influence is shown by the use of various scientific terms, such as 'τζένσο' (Smith, 1958) about the square of the unknown quantity x, and the reference to cities such as 'Φλωρέντζα', and'Ρώμη'. He is also influenced by Chinese, Hindu and Arabic sources (Chalkou, 2006).

The content of ΕλλΒιενΜαθΠραγμ.is categorized into groups as follows:

GROUP 1. Chapters 1–39, 101, 102. Operations between real numbers. Group 2. Chapters 40–56. Fractions, ratios, proportions. Group 3. Chapters 57–60. Progressions. Group 4. Chapters 61–94. Problems with first-degree equations. Group 5. Chapters 95–100, 154, 155. Interests on loan or debts. Group 6. Chapters 103–106. Division into proportionate parts. Group 7. Chapters 107–116. Problems ofsilversmithery and goldsmithery. Group 8. Chapters 117–134, 239, 240. Roots of real numbers. Group 9. Chapters 135–140. Solution of equations. Group 10. Chapters 141–153, 156–165, 234. Solution of systems. Group 11. Chapters 166–184. Plane geometry. Group 12. Chapters 202–226. Area of plane shapes. Group 13. Chapters 227–233, 235–238. Solid Geometry.

2.2. Basic results of the research. We considered it appropriate to research the probably erroneous view that Byzantium is characterized by a sterile reproduction of the knowledge of the ancient Greeks, without Byzantine scholars having made original creations. In order to support the opposite view we cite some data coming from the transcription and the mathematical commentary of more than 1000 mathematical problems of ΕλλΒιενΜαθΠραγμ.

The related research offered us valuable information about the fields of commerce, constructions, the craft of silversmithery and goldsmithery, geometrical calculations, etc. Yet what is particularly impressive is that in certain problems the methods used by the anonymous author are original and do not constitute works of reproduction or copies of works by other scientists of the time. We indicatively mention (Chalkou, 2006) the methods of calculating the sum of the terms of a progression as well as the cubic root of a number and also the unknown to us testing method of the operation of multiplication.

We also consider it important that ΕλλΒιενΜαθΠραγμ. is the first manuscript which includes the problem of inscribing a square in an equilateral triangle (Boyer, 1997) and (Jesuits, 1952), such that the base of the square constitutes part of the base of the triangle, and the other two vertices of the square belong to the other two sides of the triangle (Polya, 1998) and (Figure 3) Specifically in ΕλλΒιενΜαθΠραγμ. what is given is the side of the triangle equal to 10 and what is asked is the side of the inscribed square.

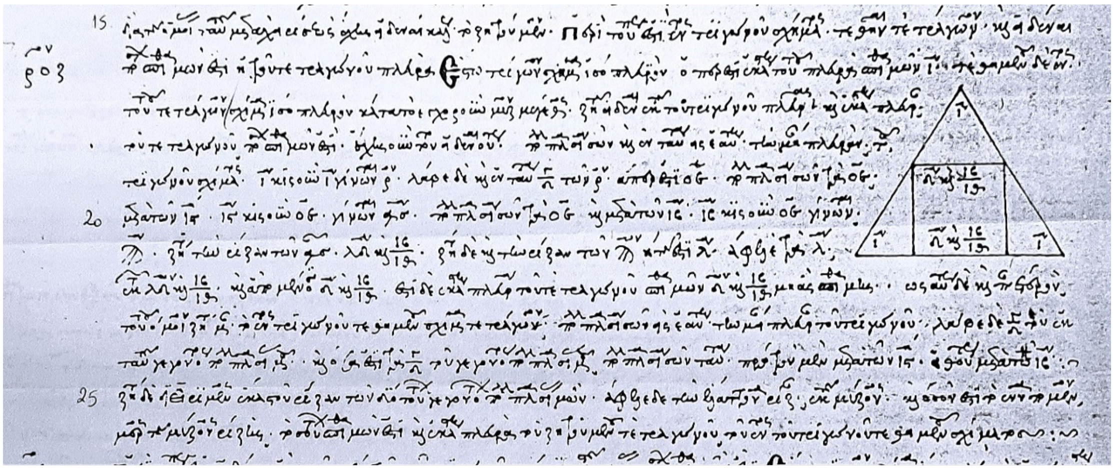

Figure 3. Inscription of a square in an equilateral triangle

Interesting is also the unknown method of calculating the volume of a solid, in the shape of a barrel, which brings to mind an attempt at an early integration (Figure 4). The author uses a mathematical formula of his own, which gives a result slightly smaller than any other known formula of his time which was then applied in the West, too (Chalkou, 2003). Specifically, we mention the case of calculating the volume of a container, in which given are the perimeters of the two circular bases equal to 10, the perimeter of the median circular section equal to 22 and the height equal to 10. The anonymous author solves the problem as follows:

He takes the ‘ἐξ ἀναλόγου’ (the average) of the perimeters of the small and the big circle, that is the average of the perimeters 10, and 22, which is equal to 16. Then he considers a new circle with a perimeter equal to 16 and calculates its area by squaring the perimeter and divides the result by 4(3+1/7).[4] Finally he multiplies this area by the height of 10 and he has the volume asked equal to 204. That is he considers that the initial container has the same volume as a container of cylindrical shape of the same height as the initial container and with a perimeter of the base equal to the average of the perimeters of the biggest and the smallest circle of the initial container.

It is worth mentioning that, when the case was generally calculating the area of a piece of land of an indistinct shape, the Byzantines applied an approximation method of their own, which was based on the perimeters of these pieces (Lefort and Bondoux and Cheynet, 1991).

As an example we consider a non-convex polygonal shape with sides 30, 8, 10, 20, 80, 2, 1, 5, 68 ‘σχοινία’. Its perimeter is equal to 224 ‘σχοινία’.

They subtracted 1 ‘σχοινίον’ for every 20 ‘σχοινία’ “because of the exaggerations and the deficits”. So they should have 224-11= 213. Instead of this they subtracted 11 from 223 and they had 212 ‘σχοινία’. Then they carried out the following calculations: 212/2=106, 106/2=53, 53.53=2809 ‘σχοινία’.

However, in order to calculate the area of regular polygons from their perimeter, which in all the cases below he considers equal to 20 spans, the author of codex 65 uses an interesting method in which he involves π, that is 3 and 1/7, (Chalkou, 2006) and (Figure 5) as follows:

[4]The anonymous author considers π=3+1/7, but does not use the specific symbol π.

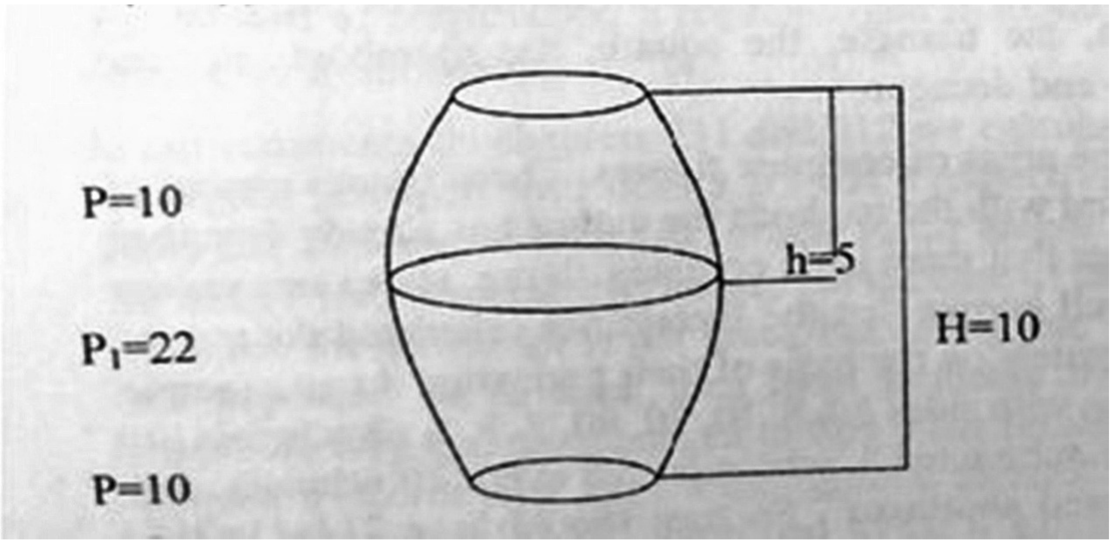

Figure 4. Calculation of the volume of a container

In the beginning in order to calculate the area of a circle with a perimeter of 20 spans he divides the square 400 of the given perimeter of 20 spans by the number 4(3+1/7)= 12 4/7. The result for the area of the circle is Ac=31 9/11.

For the regular 40-gon, he divides the square of the given perimeter (20^2=400) by the number 12 5/8, with a result A_{40}=31 7/10.

For the regular 30-gon, he divides by 12 6/9=12 2/3, with a result A_{30}=31 11/19.

For the regular 20-gon, he divides by 12 11/16, with a result A_{20}=31 1/2.

For the regular 18-gon, he divides by 12 ¾, with a result A_{18}=31 1/3.

For the regular 16-gon, he divides by 12 5/6, with a result A_{16}=31 16.

For the regular 14-gon, he divides by 12 11/12, with a result A_{14}=30 29 30.

For the regular 12-gon, he divides by 13, with a result A_{12}=30 10/13.

For the regular 10-gon, he divides by 13 1/9, with a result A_{10}=30 ½.

For the regular 8-gon, he divides by 13 8/31, with a result A_8=30 1/6.

For the regular 6-gon, he divides by 13 5/7, with a result A_6=29 1/6.

For the regular 5-gon, he divides by 14 6/13, with a result A_5=27 2/3.

For the regular 9-gon, he calculates 'τὸ ἐξ ἀναλόγου'(the average) of A_{10} and A_8.

The same for the regular 7-gon, he calculates 'τὸ ἐξ ἀναλόγου' of A_8 and A_6.

We note that the above method of calculating the area of a regular polygon is not used by the author for the shapes square and equilateral triangle.

Let it be noted that in order to calculate the areas in the cases of the 8-gon and the 6-gon we comparatively used the modern method, which is probably taught to the students of senior high school, and we discovered that the results are similar to those of the anonymous author, taking into consideration the small discrepancy because the author uses the number 3+1/7 for the number π.

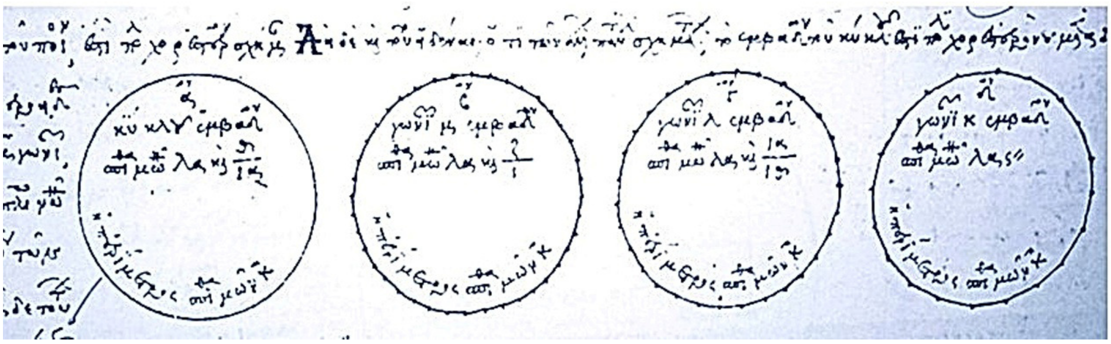

Figure 5. Calculation of the areas of regular polygons in relation to their perimeters

2.3. The first Mathematics Encyclopedia of the Byzantines (Chalkou, 2008). In the 5th Unit the author refers to the term 'μιλλιούνι', which, as it results from the definition, means a million. We know that Maximus Planoudes (c. 1260- c. 1305AD) was one of the first who used the term "milleton" (that is million) earlier. Yet according to Smith (Smith, 1958) this term first appeared in 1478 in the Italian manuscript of *Arithmetic* of Treviso. In this Italian manuscript, which was written after Codex 65, in the operation of multiplication the multiplier is placed vertically next to the multiplicand and the operation is carried out in the same way as the one used in Codex 65 (Chalkou, 2008). So, we have a significant indication that the term 'μιλλιούνι' did not first appear in the Italian *Arithmetic* of Treviso but in Codex 65, which dates from 1436 AD (Boyer and Merzbach and Kousoulakos, 1997).

In 1494 Luca Pacioli (1445- 1517AD) published *Suma* (Rose, 1975), in which he uses Hindu digits (Waerden, 2010) and calls the "cross-like method" of multiplication "crocetta" (small cross) (Tartaglia, 1556-1560 and Jayawardene, 1961).[5] *Suma de Arithmetica, Geometria, Proportioni e Proportionalita* by Luca Pacioli was the first Mathematics Encyclopedia of the Renaissance. The 1st part included Arithmetic and Algebra and the 2nd Geometry, just like Codex 65. Pacioli used as sources Fibonacci (c. 1170- c. 1240- 1250 AD), Jordanus Nemorarius (1225- 1260 AD), Blasius of Parma (c. 1350 - 1416 AD), Prosdocimo de Beldemandis (c.birth1370-1380- death 1428 AD), and Al-Khwarizmi (c. 780- c. 850AD). In the same work Pacioli, who taught arithmetic and algebra of commerce, also mentions the method of 'quadrilateral' for the multiplication of two three-digit numbers, in which the multiplier is placed in a vertical position in relation to the multiplicand (Chalkou, 2006, pp. 76-80).

Yet this is exactly how the multiplication of three-digit numbers is carried out in ΕλλΒιενΜαθΠραγμ. too, (Chalkou, 2008), which is older than *Suma* (Smith, 1958). These are some of the similar data which strengthen the view that the earlier than *Suma* ΕλλΒιενΜαθΠραγμ. is the first Mathematics Encyclopedia of the Byzantines, from which perhaps Pacioli copied, and possibly the first Encyclopedia worldwide.

Note: As a more general example regarding the multiplication in Codex 65, we refer to the multiplication of 1436 by 365 (Figure 6).

[5]The *Commercial Mathematics* by Pacioli was written in 1470- 1481 but was never published. Summa was published in 1494 and in its 600 pages it also included tables of currencies, weights and measures relevant to the cities of Italy. The work was very widespread and was taught even in the 16th c. It was the basis of N. Tartaglia's work (1556-1560) *General Trattato de Numeri et Misure*.

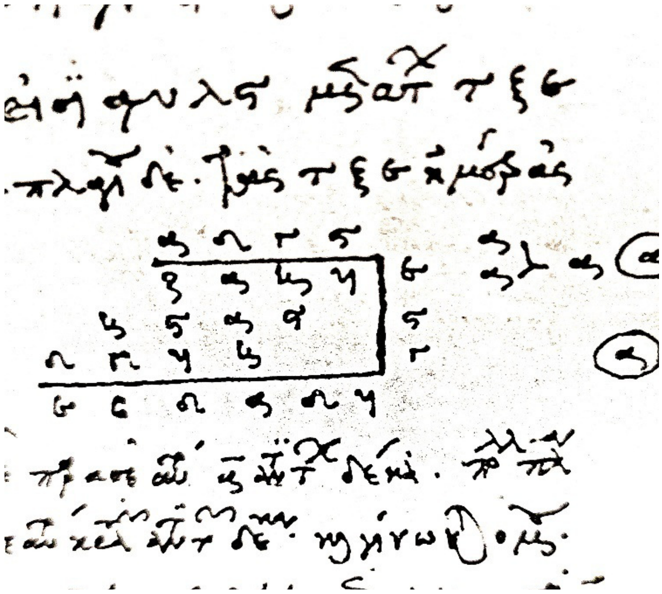

Figure 6. *The multiplier 365(γςε) is placed vertically next to the multiplicand1436 (αδγς)*

In this case the anonymous author multiplies the number ε (5) of the multiplier by the number αδγς (1436) which is the multiplicand, and writes ζαηu (7180). After that, he multiplies the next number ς (6) of the multiplier by the αδγς (1436) and writesηςας (8616).After that, he multiplies the number γ (3) of the multiplier by the number αδγς (1436), and writes δγuη (4308). Finally he adds the numbers ζαηu, ηςας, δγuηby placing them as shown in the 7th figure, and he finds the result εβδαδu (524140):

$$
\begin{array}{r|l}
\alpha\delta\gamma\varsigma & \\
\hline
\zeta\alpha\eta u & \varepsilon \\
\eta\varsigma\alpha\varsigma & \varsigma \\
\delta\gamma u\eta & \gamma \\
\hline
\varepsilon\beta\delta\alpha\delta u &
\end{array}
$$

Figure 7. Multiplying the number 1436 by the number 365 using the following symbols for the numbers from 1 to 10: 1(α), 2(β), 3(γ), 4(δ), 5(ε), 6(ς), 7(ζ), 8(η), 9(θ), 0(u).

2.4. ΕλλΒιενΜαθΠραγμ. in brief

-It contains problems of accounting (including problems of algebra) and geodesy, which correspond to today's Primary School, Junior and Senior High School, designed to be taught to students but also to state officials, merchants, builders, silversmiths and goldsmiths, etc.

- In essence it is the Mathematics Encyclopedia of the Byzantines, and perhaps the first mathematics Encyclopedia in the then known world (Chalkou, 2008).
- The term 'μιλλιούνι'did not first appear in the Italian *Suma* of Treviso in 1478 but in ΕλλΒιενΜαθΠραγμ., which dates from 1436.
- The term 'ἑκατὸν μυριάδαι' (a hundred myriads)is introduced for the million.
- It is the first Greek manuscript in which there appears the problem of drawing a square inscribed in an equilateral triangle, so that the one side of the square touches the side of the equilateral triangle (Chalkou, 2006).

- The method used by the author to test the 4 operations led us to the formulation and proof of new propositions in the field of number theory (Chalkou, 2015).
- In particular chapters it contains original for that time methods to solve problems (method of integration (Chalkou, 2003), calculating the area of regular polygons in relation to their perimeter (Chalkou, 2016), calculating the sum of terms of progressions, etc.).
- The solution of certain problems of accounting creates the impression that perhaps these constitute some early processes for measuring IQ (Chalkou, 2017).

3. Codex 72 of the 18th c. of the Public Historical Library of Demetsana

Codex 72 is one of the first works which contain non-elementary Mathematics at the time of the Ottoman Rule (Chalkou, 2009). We believe that manuscript 72 of the School of Demetsana has justifiably been considered by certain researchers to be one of the remaining 6 unpublished manuscripts of geometry and arithmetic by N. Theotokes (1731-1800 AD). We hold the view that the small differences which result from its comparison with the content of the 1st volume of the work by N. Theotokes which was published in Moscow in 1798-1799 are on the one hand due to the fact that the author of a scientific textbook which is written at a certain moment makes improvements and changes until its publication, and on the other to the fact that the manuscripts designed for teaching were naturally sometimes briefer and more simplified than the ones published in the form of a printed book (Chalkou, 2021).

Sometimes Nikephoros Theotokes published his textbooks anonymously, as for example the translation from French of the work *Testament,* to which he added wise notes (Vrokines, 1877). However, we proved with adequate data that Codex 72 is undoubtedly an intellectual creation of N. Theotokes (Chalkou, 2021).

3.1 The language, the influences, and the relevant references to the mathematical fields which comprise the content of Codex 72. We know that the language used in the school handbooks at the end of the 18th century was the simplified language of Korais (Modern Greek) (Stephanides, 1926), and Nikephoros Theotokes, who agreed with Korais, held the view that for the content of a book to be clears and easy to understand the words used must be common and understood by the people (Stephanides, 1926).

We hold the view that the language used in Codex 72 is the purist language representative of the time enriched with scientific terms, and not the affected extreme purist language, and the expressions, which are particularly taken care of so that they convey accurate, easy-to-understand and clear meanings strengthened the view that Codex 72 is indeed an intellectual creation of Nikephoros Theotokes.

Nikephoros Theotokes in his early writings preferred the demotic language of his homeland of Kerkyra (Corfu) (Great Pedagogical Encyclopedia, 1968), but towards the end of his writing activity he perfected his language (Vrokines, 1877). He was considered to be a foremost reformer of the Modern Greek language (Vrokines, 1877).

The textbooks of Theotokes gained ground because they stood out for their thorouness, clarity and the plain style of writing, but also the zeal of the teacher to transmit knowledge to his readers. N. Theotokes used works by:

1) A. Tacquet (1612-1660 AD), *Elementa Euclidea Geometricae. planae, ac solidae*… Antwerp 1654, 1665, 1672.

2) M. Ozanam (1640- 1718 AD), *Les Éléments d' Euclide Démontrés d' une manière nouvelle et facile*, Paris 1746 et 1753, and

3) Ch. Von Wolff (1679- 1754 AD), *Elementa Matheseos Universae* 1, Halle 1713 (Karas, 1992).

Theotokes includes references to Wolff in the first volume (pp. 260, 321) of his printed book which was published in Moscow in 1798-9 (Theotokes, (1798-99)). It must be noted that this first volume in essence constitutes the printed edition of Codex 72 of the library of Demetsana.

However, in the manuscript 72 itself we spotted the following names of scientists whoare mentioned and seem to have influenced its author:

P. 4a: «Εὐθεῖα γραμμὴ ἐστὶ (κατ' Ἀρχιμήδη)., or (κατὰ Πλάτωνα).., or (κατ' Εὐκλείδη)..».[6]

P. 22a: «Τῆς εὐπορωτάτης ταύτης προτάσεως ἧς ἐν πάσαις ταῖς μαθηματικαῖς ἡ χρῆσις μεγίστη, εὑρετὴν γεγονέναι φασὶ (Εὔδημος ὁ ἀρχαῖος γεωμέτρης) τὸν Πυθαγόρα. Πολλάκις δ' αὐτῆς ὁ Ἀριστοτέλης μέμνηται...».[7]

P. 85b: «Τὰ ἑξῆς θεωρήματα πλὴν τοῦ ἐσχάτου ἐκ τῶν τοῦ Πάππου εἰσὶ τοῦ Ἀλεξανδρέως».[8]

P. 150a: «..οἷον τὰ ἑλληνικὰ ἢ ρωμαϊκὰ στοιχεῖα τοῖς ἐκτεθεῖσι δὲ χρώμεθα, ὧν τινὲς μὲν Ἄραβας, τινὲς δὲ Ἰνδοὺς (ὅρα τὸ 1 σχ. τῆς τοῦ Οὐὸλφ Ἀριθμητικῆς)».[9]

P. 164b: « (Διοφάντου ἐν τῷ 1ῳ καὶ 2ῳ ὁρισμῷ τοῦ 1ου βιβλίου) ».[10]

P. 175b: «Τῶν λογαρίθμων οἷς οἱ μαθηματικοὶ χρῶνται τὸν τύπον ὁ Βρίγγιος τῇ τοῦ πρώτου αὐτῶν εὑρετοῦ Νεπέρτου ὁδηγίᾳ πρῶτος συνετάξατο..».[11]

The first part of Codex 72, that is the one which includes the Euclidean geometry, contains the 1st book of Euclid, in which there are the basic geometrical definitions (of the point, straight line, angle, circle, triangle, parallelogram, etc.) and 48 propositions are proven.

The 2nd book deals with the parallelograms and their areas.

The 3rd book deals with the circle.

In the 4th book there is the theoretical approach to the inscribed and circumscribed polygons in a circle.

The 5th book contains the ratios and the proportions.

In the 6th book the author presents the similar rectilinear shapes.

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6 «A straight line is according to Archimedes.., or according to Plato..., or according to Euclid..». Archimedes (c. 287- c. 212 BC), Plato (c. 425-c.348 BC), Euclid (fl. 300 BC).

7 «Pythagoras is said (by Eudimus the ancient geometer) to be the founder of this great proposition, whose usefulness in all topics related to Mathematics is great. Aristotle invokes it many times...». Pythagoras (c. 570- c. 495 BC), Aristotle (c.384- c.322 BC), Eudemus of Rhodes (c. 370- c. 300BC).

8 «The following theorems, except the last one, belong to Pappus of Alexandria». Pappus of Alexandria (c. 290- c. 350 AD).

9 «…We need only the Greek, or the Roman figures of those who communicated them to us, some of whom were Arabs, and some were Hindu (see the 1st comment on Wolff's Arithmetic)». Christian Wolff (1674- 1754 AD).

10 «In the 1st and 2nd definitions of the 1st book by Diophantus». Diophantus of Alexandria (c. 210- c. 290 AD).

11 «Henry Briggs was the first to construct the form of logarithms used by mathematicians under the instruction of John Napier, who invented it in the first place». Henry Briggs (1561-1630AD), John Napier (1550- 1617 AD).

The 11th book, or the 1st of solids, refers to solid geometry, from the basic definitions to the parallelepipeds.

The 12th book examines issues related to the pyramid, the cone, the cylinder as well as the solid shapes inscribed or circumscribed in them.

The second part of Codex 72 contains 5 books, which include issues of arithmetic related to the 1st book with the definitions of the numbers and the operations between them, the Pythagorean table and the tests of the operations.

The 2nd book contains the fractions and the decimal numbers with their operations.

The 3rd book contains methods of calculating square and cubic roots of real numbers.

In the 4th book presented are issues related to logarithms and progressions, and

The 5th book refers to practical problems of everyday life mainly concerning commerce.

In the 4th book Theotokes, like Napier, uses the progressions to find the theory of logarithms, according to which "Logarithms are numbers which correspond to the analogous numbers (terms of a geometric progression) and have equal differences (terms of an arithmetic progression)" (Nepero, 1614) and (Chalkou, 2015).[12]

3.2 Results of the research about the influence of Byzantine Mathematics on Codex 72. In the manuscript 72 the million is referred as 'μιλλιόνιο' and the billion as 'διλλιόνιο'. It must be noted that in the 5th Chapter of the manuscript 65 its author mentions the term 'μιλλιούνι', which, as it results from the definition, means the million. In the 3rd book of the arithmetic of the manuscript 72 (1st definition) we read: «Ἐξ ἀριθμοῦ τινὸς ἐφ' ἑαυτὸν πολλαπλασιασθέντος ὁ γινόμενος, τετράγωνος λέγεται. Οὗτος δὲ ὁ ἀριθμός, πλευρὰ ἢ ρίζα τετραγωνική». Respectively in the 2nd definition: «Ἐκ τετραγώνου ἐπὶ τῆς ἑαυτοῦ ρίζης πολλαπλασιασθέντος ὁ γινόμενος, κύβος καλεῖται τοῦ ἀπ' ἀρχῆς ληφθέντος, ὁ δὲ ἀπ' ἀρχῆς ληφθείς, κυβικὴ ρίζα».[13]

In the ΕλλΒιενΜαθΠραγμ. of 1436 AD the respective definitions are 'κορυφή', and 'ρίζακουβική' or 'τετράγωνος'.

In Chapter 239 the author give tables for calculating roots for certain numbers from 1 to 1000.

In the ΕλλΒιενΜαθΠραγμ. there are no mathematical formulas but guidance for calculating the roots. Although Diophantus had introduced his own symbolism already in 275 AD, no use of it is made, a fact which we observe in the manuscript, too.

In the 4th book of the Arithmetic of the manuscript 72 (6th definition) the author writes: «Αἱ ἐκ πολλῶν ὅρων συνιστάμεναι ἀριθμητικαὶ ἀναλογίαι σειραὶ ἢ πρόοδοι

[12]Logarithms were discovered by the Scottish mathematician John Napier in 1614. In 1615 Henry Briggs suggested that Napier should use the base of 10. Napier died in 1617 and Briggs continued alone, publishing in 1624 a table of logarithms calculated to 14 decimal places for numbers from 1 to 20,000 and from 90,000 to 100,000. The Dutch publisher Adriaan Vlacq(1600- 1667 AD) brought out a 10-place table for values from 1 to 100,000, adding the missing 70,000 values. The gap was filled in by Adriaan Vlacq in this work which was published in 1628 in Gouda of Holland (Great American Encyclopedia, 1976), (Encyclopedia Britannica, logarithms, 2024) and (Baralis, 1994).

[13]«If a number is multiplied by itself, it is called 'square'. The initial number is called 'side', or 'square root'».

«If a square number is multiplied by its root, then their product is called the 'cube' of the initial number, and the initial number is called the 'cubic root'».

λέγονται ἀριθμητικαί, (ἐν αἷς ἡ 1,2,3,4,5,6,7 κατὰ σειρὰ φυσικὴ ἀριθμητική), ὁμοίως καὶ ἐκ πολλῶν γεωμετρικῶν ὅρων, γεωμετρικαί».[14]

At the end of the edition of *Arithmetic Introduction* (1866) by Nicomaque de Gérase (60-c. 120 AD), certain problems are included, which are attributed to a certain monk named Isaac and refer to calculating the terms of an arithmetical progression. These are problems similar to those in the ΕλλΒιενΜαθΠραγμ. and their suggested solutions are also of the same type as the ones in our manuscript, although in it the terms arithmetical and geometrical progression are not used.

The author of the Codex 72 defines 'the rule of three' as follows: ««Ὁ τριῶν δοθέντων ὅρων τὸ τέταρτον ὁδηγούμενος εὑρίσκειν κανών»[15]. In the ΕλλΒιενΜαθΠραγμ. this method is often used under the name 'ἡ διὰ τῶν τριῶν μεταχείρισις' (the rule of three) and is based on the properties of proportions. In Chapter 53 the author writes: «Διὰ γὰρ τῶν γ καὶ δ καὶ ε, οἵτινες εἰσὶ τρεῖς ἀριθμοὶ ἀνόμοιοι, εὑρίσκεται τὸ ζητούμενον. Ἐνεργοῦντες δὲ οἱ τρεῖς τὸ ζητούμενον, καλῶς ἂν καὶ διὰ τῶν τριῶν λέγεται».[16]

The author of the Codex 72 uses the term 'method of the company' in problems where what is asked is the calculation of the sum of the profit to which each partner of the business is entitled. In the Codex 65 the problems of distribution and company are solved with a method original for what is scientifically known so far. The innovation of the Codex 65 is the way of distributing the profit of a company to its partners, where the initial capital is added instead of being multiplied by the time that it remains in the business.

In both manuscripts the method of 'false hypothesis' is used. The author of the Codex 72 describes it as follows: «Ψευδοῦς ὑποθέσεως μέθοδος λέγεται, διὰ τὸ ἐξ ὑποθετικῶν καὶ ἀναλόγων μὲν τοῖς ζητουμένοις, μὴ ἀληθῶν δὲ ἀριθμῶν τὴν τῶν προβλημάτων προκύπτειν ἐπίλυσιν».[17]

In propositions 47 and 48 of the 1st book of geometry of the manuscript 72 the author proves the Pythagorean Theorem yet without giving a name to it. In the Codex 65 the Pythagorean Theorem is called 'κανών τῆς σκάδρας' (the rule of scadra). Also, in the 2nd proposition of the 12th book of the manuscript 72 the author mentions: «Οἱ κύκλοι πρὸς ἀλλήλους εἰσὶν ὡς τὰ ὑπὸ τῶν διαμέτρων τετράγωνα», and «Αἱ τῶν κύκλων περιφέρειαι πρὸς ἀλλήλας εἰσὶν ὡς αἱ διάμετροι»..[18] We notice that in both manuscripts the value of this constant ratio is not symbolized with the letter π.

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[14] «The arithmetic proportions, series, or progressions, which consist of many terms are called 'arithmetical progressions' (among which 1,2,3,4,5,6,7,,.. which is called 'natural arithmetical'); similarly those consisting of many geometric terms, are called 'geometrical progressions'».

[15] «The rule in order to find the fourth term when the three terms are given».

[16] «From 3 and 4 and 5, which are three different numbers, you find the one being asked for. That is why the method is called 'the rule of three'».

[17] «The method is called the one 'of false hypothesis' because using hypothetical numbers, proportionate to the ones being asked for, but not real, you solve the problems».

[18] «The relationship between the circles is as the relationship between the squares of their diameters», and «The relationship between the circumferences of the circles is as the relationship between their diameters».

3.3 In brief about the Codex 72

- Nikephoros Theotokes was one of the first who tried to combine the Ancient Greek science with the knowledge coming from the West. This led to highlighting the philosophical work of the Greek scientists as the main criterion for evaluating of the civilization of that time in Modern Greek History.
- It is one of the first texts with non-elementary Mathematics at the time of the Ottoman Rule.
- From the correspondence between the Teachers of the Greek nation we conclude that it was taught in various schools of the time and was held in high esteem.
- The adoption of Mathematics and Physics as primary subjects in the school curriculum in Greece before the War of Independence is attributed to Nikephoros Theotokes.
- Nikephoros Theotokes was considered to be the main reformer of the Modern Greek language, which «ἐκκαθάρας ἐπλούτησε διὰ τοῦ ἐμμουσοτάτου καὶ πολυτερπεστάτου ἀρχαϊκοῦ κάλλους»[19] (Vrokines 1877).
- Although it is a work which contains modern Mathematics of the 18th c., still it keeps the influence of Byzantine Mathematics.

4. Conclusions

The two aforementioned Greek manuscripts of the 15th and the 18th c., the ΕλλΒιενΜαθΠραγμ. and the Codex 72, gave us the opportunity to deal with the teaching of Mathematics during the Byzantine Empire and then the Ottoman Rule. These are two time periods where because of lack of sufficient sources related to our Cultural Heritage the information was too little for us to form a view on Mathematics in the Greek schools. The above mentioned two codices enriched our knowledge to a fairly satisfactory degree.

The ΕλλΒιενΜαθΠραγμ. led us to formulate new propositions in number theory and knowledge of new methods in various fields in the science of Mathematics, as well as forming the view that it constitutes the first Encyclopedia of Mathematics of the Byzantines, with a possibility that it was the first Encyclopedia of Mathematics worldwide.

The Codex 72 offered us information about the level of teaching of Mathematics in certain schools, which -as the School of Demetsana - were of University level. Apart from the Euclidian geometry the Codex contains algebra, but also practical arithmetic, which includes problems solved with methods identical to those used by the Byzantines, thus proving the uninterrupted continuity in the teaching of Mathematics in our country.

It is reasonable to conclude that it would be desirable to have at our disposal more manuscripts with Mathematics from the dozens which are found in the libraries of Greece and abroad. Let us not forget that the Codex 65 was found about 1995 in the National Library of Austria in Vienna thanks to the research by Professor Pantelis Karelos, and the Codex 72 in 1996 during my tour in the Public Historical Library of Demetsana.

It would be desirable for researchers to be able to search more easily for digitized manuscripts with a view to publication – valuable elements of our Cultural Heritage – which would be immediately available for study.

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[19] «once he had cleared the language, he enriched it with the melodious and marvellous ancient beauty».

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mchalkou@gmail.com